J. DIFFERENTIAL GEOMETRY
45 (1997) 611-637

SYMPLECTIC DEFORMATIONS OF CALABI–YAU THREEFOLDS

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Introduction

Much recent work on Calabi–Yau threefolds has concentrated on the analogy between small deformations of the complex structure on a Calabi–Yau threefold X, which by the unobstructedness theorem of Bogomolov, Tian and Todorov are parametrized by a neighbourhood of the origin in $H^1(T_X) = H^{2,1}(X, \mathbf{C})$, and small deformations of the symplectic structure, which by Moser's Theorem are parametrized by a neighbourhood of the origin in $H^2(X, \mathbf{R}) = H^{1,1}(X, \mathbf{R})$. This analogy is described in [8] in the context of Mirror Symmetry.

We recall that a Calabi–Yau threefold has various differential invariants, in particular the cubic form on $\mathrm{H}^2(X, \mathbf{Z})$ given by cup-product, the linear form on $\mathrm{H}^2(X, \mathbf{Z})$ given by cup-product with the first Pontryagin class p_1 (which in the Calabi–Yau case is just the second Chern class c_2), and the middle cohomology $\mathrm{H}^3(X, \mathbf{Z})$. Indeed, if X is simply connected and has no torsion in $\mathrm{H}^3(X, \mathbf{Z})$, these invariants specify the differential class of the manifold precisely [19].

In this paper, we shall denote a Calabi–Yau threefold by a pair (X, J), where X is the differential manifold and J is the (integrable) complex structure. A symplectic form ω on X which is compatible with J is just a Kähler form on the complex manifold. Moreover, the almost complex structures which are compatible with a given symplectic form on X form a contractible set, as do the almost complex structures tamed

Received September 18, 1995, and, in revised form, February 16, 1996.
Mathematics Subject Classification (1991): 14J10, 14J15, 14J30, 32J17, 32J27, 53C15, 53C23, 57R15, 58F05.

by the symplectic form. It follows therefore that if we have a family of symplectic forms ω_t on X, we can find a corresponding family J_t of almost complex structures, with J_t tamed by ω_t. We are led therefore to considering *almost complex* deformations of J – for very accessible surveys of the concepts from Symplectic Geometry we need, the reader is referred to [10], [11]. It will follow from our results that if two Calabi–Yau threefolds (with given Kähler forms) are symplectic deformations of each other, they will (from the Kähler point of view) have the same kind of properties in common as if they were algebraic deformations of each other.

More specifically, this paper represents an extension of the author's previous results [21] concerning the deformation properties of the cone of Kähler classes $\mathcal{K} \subset \mathrm{H}^2(X, \mathbf{Z})$ under deformations of the complex structure (it may also be viewed as an extension of the work in [15]). Perhaps surprisingly, these results generalize in an almost unchanged form to the almost complex case. This involves first defining the Kähler cone for the almost complex case in terms of pseudo-holomorphic rational curves, and then generalizing our previous theorem to include almost complex deformations of (X, J).

For an almost complex structure J' on X, we consider the open cone $H^2(X, \mathbf{R})$ consisting of classes D with $D^3 > 0$ and such that $D \cdot A > 0$ for all homology classes A represented by pseudo-holomorphic rational curves $f : \mathbf{CP}^1 \to X$. We *define* the Kähler cone $\mathcal{K} \subset H^2(X, \mathbf{R})$ of (X, J') to be the connected component of this cone which contains the classes of symplectic forms taming J' (which we note form a convex set). By results of [20], [21], this corresponds to the usual Kähler cone when (X, J') is a Calabi–Yau threefold with integrable complex structure.

We shall say that a conic bundle E over a smooth curve C is a *quasi-ruled* surface over C if the normalization $\tilde{E}$ is a $\mathbf{P}^1$-bundle over an *unramified* cover $\tilde{C}$ of C – thus E is either a ruled surface over C or a conic bundle all of whose fibres are line pairs. The main result from [21] said that the Kähler cone jumped up under generic complex deformations if the Calabi–Yau threefold contained quasi-ruled surfaces over elliptic curves, and otherwise was invariant. Adopting terminology from [22], the Kähler cone of a generic *holomorphic* deformation of a Calabi–Yau threefold (X, J) will be called the *general Kähler cone* of (X, J), and will equal the Kähler cone unless (X, J) contains quasi-ruled surfaces over an elliptic curve. When (X, J) does contain elliptic quasi-ruled surfaces, the Kähler cone is the subcone of this general Kähler cone cut out by the further inequalities $D \cdot l > 0$ for l any fibre of one of the elliptic quasi-ruled surfaces; in [22], this is proved when (X, J) contains just elliptic ruled surfaces, but the proof generalizes easily.

Theorem 1. *Suppose that (X, J) is a Calabi–Yau threefold containing no quasi-ruled surface over an elliptic curve. Let ω be a Kähler form on (X, J) and let $\mathcal{J}(\omega)$ denote the space of almost complex structures tamed by ω. Then the Kähler cone of (X, J') at any point $J' \in \mathcal{J}(\omega)$ is a subcone of the Kähler cone of (X, J).*

Remarks. (i) Since any given Kähler form ω' on (X, J) will tame almost complex structures $J' \in \mathcal{J}(\omega)$ sufficiently close to J, it follows that the Kähler cone $\mathcal{K}$ at J is a limit of the Kähler cones of (X, J') as J' tends to J. The Kähler cone therefore depends continuously on the almost complex structure at any integrable point J with (X, J) containing no elliptic quasi-ruled surfaces.

(ii) The reason why we cannot make the stronger claim that $\mathcal{K}$ equals the Kähler cone at the generic point $J' \in \mathcal{J}(\omega)$ is that it is at least theoretically possible that there exists a homology class A (with zero Gromov-Witten invariants) which is J'-effective for J' in the closure of some non-empty open subset of $\mathcal{J}(\omega)$, but in some open neighbourhood of J is not J'-effective. Here we are using the terminolgy from [11], where A is called J'-effective if it can be written as a positive integral combination of classes which can be represented by pseudo-holomorphic *rational* curves, and we are implicitly using Gromov compactness, as stated for instance in [18, Theorem 5.2.3] or [11, Theorem 4.4.3.].

We can however generalize Theorem 1 to allow deformations of the symplectic form. Let ω_t be any family of symplectic forms on X with ω_0 a Kähler form on (X, J). We can then find a family of almost complex structures J_t on X with $J_0 = J$ such that J_t is always ω_t tamed. If (X, J) contains no elliptic quasi-ruled surfaces, the Main Claim from §1 implies that the Kähler cone of (X, J_1) is a subcone of the Kähler cone $\mathcal{K}$ of (X, J).

Theorem 2. *Two Calabi–Yau threefolds which are deformations of each other as symplectic manifolds will have the same general Kähler cone.*

This follows immediately, since either general Kähler cone is a subcone of the other. There is however an example due to Mark Gross of two diffeomorphic Calabi–Yau threefolds (both generic in complex moduli) for which one Kähler cone is a proper subset of the other, and hence are not symplectic deformations of each other [5], [16]. Thus each cohomology class which lies in both Kähler cones must intersect at least two connected components of symplectic forms, and the Kähler forms of the two threefolds must lie in different components.

We can furthermore identify the classes of symplectic forms from the same connected component as the Kähler forms on (X, J).

Theorem 3. *Assuming that the Calabi–Yau threefold* (X, J) *contains no elliptic quasi-ruled surfaces, the Kähler cone consists precisely of the classes of symplectic forms in the same connected component as the Kähler forms.*

If now we consider the almost complex structures on X which are tameable (by some non-specified symplectic form) and which give rise to trivial first Chern class, we obtain a space $\mathcal{J}_0$ containing the complex moduli spaces of all integrable Calabi–Yau structures on X. Our results show that two such integrable Calabi–Yau structures have surprising similarities if they lie in the same connected component of $\mathcal{J}_0$. Moreover, since such structures in the same connected component of $\mathcal{J}_0$ will have the same Kähler cone, we can choose a polarization common to all of them, and then from standard Hilbert scheme theory there can be at most finitely many complex families contained in the given component. I conjecture that only finitely many connected components of $\mathcal{J}_0$ will contain Kähler structures, and hence (equivalently) that there are only finitely many families of Calabi–Yau threefolds for each diffeomorphism class. I do not in fact know of any examples where more than one family is contained in a given connected component of $\mathcal{J}_0$, and so the possibility cannot yet be ruled out that two Calabi–Yau threefolds which are symplectic deformations of each other might always lie in the same algebraic family.

The structure of this paper is that our results are reduced to a single Main Claim in the first Section, which in turn reduces to showing that certain Gromov-Witten invariants are non-zero. It might be added here that the specific results from Symplectic Geometry that we need appear in the preprints [16], [17], but that we shall also quote the more expository text [11]. The Main Claim is checked immediately in the case of faces of the Kähler cone corresponding to small contractions. The remaining Sections are devoted to proving the Main Claim in the case of non-small contractions, for which we shall also need a more precise classification of birational contractions on Calabi–Yau threefolds.

1. Reduction to Main Claim

For (X, J) a complex Calabi–Yau threefold, we denote by $\overline{\mathcal{K}}$ the closure of the Kähler cone $\mathcal{K} \subset H^2(X, \mathbf{R})$, which can be interpreted as consisisting of the real divisor classes D which are *nef*, that is $D \cdot C \geq 0$ for all curves C on X. We let $W^* \subset H^2(X, \mathbf{R})$ denote the cubic cone of real divisor classes D with $D^3 = 0$. We recall from the results of [21] that $\overline{\mathcal{K}}$ is locally finite rational polyhedral away from W^*, and that the codimension-one faces (not contained in W^*) of $\overline{\mathcal{K}}$ correspond to primitive birational contractions $\phi : X \to \bar{X}$ of one of three types. Type I is when ϕ contracts down a finite number of (numerically equivalent) curves, all smooth and rational. Type II is when ϕ contracts down an irreducible surface E to a singular point - here, E will be a generalized del Pezzo surface (see §2 for more details). Type III is when ϕ contracts down an irreducible surface E to a curve C of Du Val singularities, generically either an $c\mathrm{A}_1$ or $c\mathrm{A}_2$ singularity, and E is a conic bundle over C (see §3 for more details). In each case, there are rational curves l contracted by ϕ, and for any one of these the face is determined by $\{D \in \overline{\mathcal{K}} : D \cdot l = 0\}$. Thus the Kähler cone as *defined* in the Introduction in the almost complex case is a natural generalization from the integrable case.

Main Claim. *Suppose (X, J) is a Calabi–Yau threefold. For every codimension-one face of the nef cone $\overline{\mathcal{K}}$ not corresponding to the contraction of an elliptic quasi-ruled surface to a curve of singularities and not contained in the cubic cone, we can find a homology class $A \in H_2(X, \mathbf{Z})$ which defines the face and is J_1-effective for any J_1 tamed by a symplectic form ω_1 in the same connected component as the Kähler forms on (X, J).*

In the case where the face does correspond to the contraction of an elliptic quasi-ruled surface, it follows from [21] that there exists a class A represented by a smooth rational curve on (X, J), but not effective for a general holomorphic deformation.

The Main Claim tells us that if (X, J) is general in complex moduli, the Kähler cone cannot get larger under a tamed almost complex deformation, and hence Theorems 1 and 2 from the Introduction follow. Theorem 3 also follows easily; the non-trivial claim here is that for any symplectic form ω_1 in the same connected component as the Kähler forms on (X, J), the class of ω_1 lies in the the Kähler cone $\mathcal{K}$ of (X, J). Let ω_t be a family of symplectic forms on X with ω_0 a Kähler form on

(X, J) and ω_1 the given symplectic form. As noted before, we find a family of almost complex structures J_t on X with $J_0 = J$ such that J_t is always ω_t tamed. If (X, J) contains no elliptic quasi-ruled surfaces, the Main Claim implies that the Kähler cone of (X, J_1) is a subcone of the Kähler cone $\mathcal{K}$ of (X, J); however the class of ω_1 clearly lies in the Kähler cone of (X, J_1), and so the result follows. The results stated in the Introduction all reduce therefore to the Main Claim as stated above.

Let us prove the Main Claim in the case of Type I contractions. This is the simplest of the three cases, but will illustrate the general principle that in most cases we can reduce down to considering local *holomorphic* deformations. We shall explain the technical results needed from Symplectic Geometry reasonably fully in this case, and will refer back to the proof below when considering the other two cases.

Proposition 1.1. *If $\phi : X \to \bar{X}$ is a primitive Type I contraction, then for each component Z of the exceptional locus, there exists a neighbourhood U of Z and a holomorphic deformation of the complex structure on U under which Z splits up into disjoint $(-1,-1)$-curves.*

Proof. The morphism ϕ contracts Z to a single cDV (compound Du Val) singularity. Taking a Stein neighbourhood $\bar{U}_0$ of this singularity, there exists by the argument on p. 679 of [4] a flat 1-parameter family of isolated threefold singularities $\bar{\mathcal{U}} \to \Delta$, where $\bar{U}_t$ (for $t \neq 0$) contains precisely $\delta > 0$ nodes as its singular locus. Moreover, we may assume also that there is a simultaneous resolution $\mathcal{U} \to \Delta$ with U_0 being a neighbourhood of Z in $X = X_0$, and U_t (for $t \neq 0$) containing precisely δ disjoint $(-1,-1)$-curves. We can however take a *good representative* of our germ $\bar{\mathcal{U}} \to \Delta$ such that on the boundary the family is differentiably trivial (see Theorem 2.8 of [9]). If we take $\mathcal{U}' \to \Delta$ to be the corresponding family of resolutions (a family of manifolds with boundary, where the boundary is differentiably trivial over Δ), the Ehresmann Fibration Theorem (for manifolds with boundary) then applies to show that $\mathcal{U}' \to \Delta$ is itself differentiably trivial. Therefore, the family $\mathcal{U} \to \Delta$ of resolutions may be taken to be differentiably trivial, and may as a result be regarded as a family of holomorphic deformations of the complex structure on the (fixed) neighbourhood U of Z in X.

We now use the first feature of almost complex structures which is not true in the holomorphic case - that they may be patched together in a C^∞ way by a partition of unity argument. Thus, if we take neighbourhoods of each connected component of the exceptional locus of a primitive Type I contraction and deform the complex structure locally so that each component splits up into disjoint $(-1,-1)$-curves, we can patch these local (integrable) structures together with the original complex structure to yield an almost complex structure J' which is integrable in a neighbourhood of each $(-1,-1)$-curve appearing. Each such curve will have homology class lying in the (extremal) 1-dimensional ray determined by the codimension-one face.

Let us fix a hyperplane class H of (X,J) and let ω denote a corresponding Kähler form. If we let A be the homology class represented by one of these $(-1,-1)$-curves of minimal degree with respect to the given Kähler form ω, then A is J-effective, but irreducibly so, since it is only represented by irreducible embedded holomorphic rational curves. From Gromov compactness, we deduce that J' may be chosen above so that A is not only J'-effective but also J'-indecomposable, i.e., A is only represented by simple pseudo-holomorphic rational curves on (X,J'). Moreover, we may assume that the pseudo-holomorphic rational curves found above are the only possible ones representing A. This latter statement also follows from Gromov compactness, since by considering the compact subset F of X given by the complement of suitably small open neighbourhoods of the connected components of the exceptional locus, we may assume that any other pseudo-holomorphic rational curve on (X,J') representing A intersects F. If this were true for all the almost complex structures J' constructed in the above way, we would deduce from Gromov compactness that there was a holomorphic curve on (X,J) representing A and intersecting F, which is nonsense.

Proposition 1.2. *The Main Claim is true for primitive Type I contractions.*

Proof. Choose A and J' as above, with J' sufficiently near J that it too is ω-tamed (this being an open condition). A standard regularity criterion (see [10, (4.2)] or [11, (3.5.1)]) then implies that the almost complex structure J' is regular (for the class A) – note, that the proof given only requires J' to be integrable in a neighbourhood of each pseudo-holomorphic rational curve representing A.

Since (X,J) is Calabi–Yau, the symplectic manifold (X,ω) is weakly monotone, and any almost complex structure in $\mathcal{J}(X,\omega)$ is semi-positive. Thus if A is a non-multiple class, the theory of [11, §7.2], gives a well-defined 3-point Gromov-Witten invariant $\Phi_A(H,H,H)$, invariant also under symplectic deformations of ω. Moreover, the property that J' is integrable in a neighbourhood of each pseudo-holomorphic rational curve representing A implies that J' induces an (integrable) complex structure on the moduli space $\mathcal{M}(A, J')$ (see [11, Remark 3.3.6]), and hence the natural orientation at the points in the moduli space of unparametrized curves representing A is always positive (cf [11, Remark 7.3.6]). Thus $\Phi_A(H, H, H) = \Phi_{A,J'}(H, H, H)$ is equal to a positive multiple of $(A \cdot H)^3$.

When A is a multiple class, i.e., $A = mB$ for some $m > 1$, the Gromov -Witten invariant Φ_A is not well-defined for the reasons explained in [11, §9.1]. To circumvent this difficulty, we follow the original idea of Gromov and work with the graphs of pseudo-holomorphic rational curves $\mathbf{CP}^1 \to X$. We let $\hat{X}$ denote the product manifold $\mathbf{CP}^1 \times X$, with $\hat{\omega}$ the product symplectic form $\tau_0 \times \omega$ where τ_0 is the standard symplectic form on $\mathbf{CP}^1$ corresponding to the Fubini-Study metric, and $\hat{J}'$ the product almost complex structure $i \times J'$ with i the complex structure on $\mathbf{CP}^1$. Let A_0 denote the homology class $[\mathbf{CP}^1 \times \{pt\}] \in H_2(\hat{X}, \mathbf{Z})$, and let $\hat{A}$ denote the class $A_0 + A \in H_2(\hat{X}, \mathbf{Z})$. As explained in [11, §9.1], we now work on $\hat{X}$ rather than X, counting $\hat{A}$-curves on $\hat{X}$ rather than A-curves on X; the advantage of this is that the class $\hat{A}$ is necessarily primitive (i.e., non-multiple), whilst the same was not true for A. For a general $J'' \in \mathcal{J}(\omega)$, we must then work, not with the product almost complex structure, but with a small perturbation.

The theory described in [11] does not quite cover the Calabi–Yau case, and so instead we refer to [16]. The invariant we need is called $\tilde{\Phi}$ there, and its construction and properties are described fully in §3 of the paper – it is in fact also a special case of the mixed invariant Φ constructed in [17], which could therefore serve as an alternative reference. To define $\tilde{\Phi}$, Ruan follows the idea of Gromov and considers a special type of small perturbation J_g from the product almost complex stucture on $\mathbf{CP}^1 \times X$, where g is an anti-complex linear bundle map from $T\mathbf{CP}^1$ to TX. With this structure, a map $f : \mathbf{CP}^1 \to X$ satisfies the equation $\bar{\partial}_{J''} f = g$ (which should be regarded as a small perturbation from the condition $\bar{\partial}_{J''} f = 0$ needed for f to be holomorphic) if and only if the corresponding section $\bar{f} : \mathbf{CP}^1 \to \mathbf{CP}^1 \times X$ is J_g-holomorphic.

In this way, Ruan is able to construct a well-defined invariant $\tilde{\Phi}_{(A,\omega)}$ by "counting the number of perturbed holomorphic maps with marked points" (see [16, (3.3.6)]), which in our case will also be invariant under symplectic deformations of ω (see Lemma 3.3.4 and note that semi-positivity is satisfied in our case). The fact that the invariant is well-defined essentially follows from the fact that $\hat{A}$ is a primitive (i.e., non-multiple) class and a standard cobordism argument. We shall in this paper therefore denote this invariant by $\tilde{\Phi}_A$, and refer to $\tilde{\Phi}_A(H,H,H)$ as the three point Gromov-Witten invariant – it might be added however that the literature in this field employs many different terminologies and notation. In the case where A is primitive, by Proposition 3.3.8 of [16], the two definitions yield the same three-point invariant.

In order to calculate this invariant in our particular case, we observe that the pair $(J', 0)$ is A-good, in the terminology of [16, (3.3.7)] – this follows from the A-regularity of J' observed before. Thus we are able to calculate $\tilde{\Phi}_A(H,H,H)$ from the almost complex structure J'. In fact, we have a precise knowledge of the moduli space $\mathcal{M}_{(A,J',0)}$ of parametrized pseudo-holomorphic rational curves representing A, and if we consider as in [16] marked pseudo-holomorphic rational curves (with three marked points on $\mathbf{CP}^1$) then we obtain a finite set. This enables us to calculate that $\tilde{\Phi}_A(H,H,H) = \tilde{\Phi}_{(A,J',0)}(H,H,H)$ is equal to some positive multiple of $(A \cdot H)^3$ as before.

Thus, for small perturbations $J_{1,g}$ of the product almost complex structure $i \times J_1$, there are pseudo-holomorphic rational curves representing the class $\hat{A} = A_0 + A$. By Gromov compactness, we deduce for the product structure that the the class $\hat{A}$ is represented by a pseudo-holomorphic rational curve or cusp curve, and hence projecting onto the first factor that the class A is J_1-effective. The proof of the Main Claim is now complete for Type I contractions.

The above argument provides the model also for the proof of the Main Claim in the case of contractions of Types II and III; we find a class A defining the relevant face of $\overline{\mathcal{K}}$ for which $\tilde{\Phi}_A(H,H,H)$ can be shown non-zero, and hence the Main Claim will follow.

2. Type II contractions

Suppose now that $\phi : X \to \bar{X}$ is a primitive Type II contraction, therefore contracting a generalized del Pezzo surface E down to a singular point. As was the case for Type I contractions, the technique we use to prove the Main Claim is to make a small holomorphic deformation on some open neighbourhood of the exceptional locus, under which E deforms to a smooth del Pezzo surface – the fact that this is always possible has been proved by Gross in [6]. We then patch together with the original complex structure, and observe that in most cases there is an obvious choice for the class A. Where this is not the case, we have to use a cobordism argument of Ruan.

In passing, I remark that it is an interesting question whether there is a global holomorphic deformation of X under which E deforms to a smooth del Pezzo surface. This is unknown and is the analogous problem to a conjecture of Clemens for Type I contractions that there is a global holomorphic deformation under which the exceptional locus splits up into distinct $(-1,-1)$-curves – cf. Proposition 1.1 where we prove that this does happen for deformations on some open neighbourhood of the exceptional locus.

Generalized del Pezzo surfaces may be classified, but as we see below, not all of them occur as the exceptional surface of a Type II contraction. The normal del Pezzo surfaces are either elliptic cones or del Pezzo surfaces with rational double point singularities as classified in [3] – for a proof of this statement, see for instance [7]. The non-normal del Pezzo surfaces have been classified in [14]. Let us first consider the possibilities for small values of E^3.

If $E^3 = 1$, the singularity of $\bar{X}$ is a hypersurface singularity with equation of the form $x^2+y^3+f(y,z,t)=0$ where $f = yf_1(z,t)+f_2(z,t)$ and f_1 (respectively f_2) is a sum of monomials of degree at least 4 (respectively 6). The morphism $\phi : X \to \bar{X}$ is then the weighted blow up of the singularity with weighting α given by $\alpha(x)=3$, $\alpha(y)=2$ and $\alpha(z)=\alpha(t)=1$ (for a proof of these statements, see [12, (2.10) and (2.11)]). Thus E is naturally embedded in $\mathbf{P}(3,2,1,1)$ with equation $x^2+y^3+yg_1(z,t)+g_2(z,t)=0$, where g_1 (resp. g_2) is the degree 4 (resp. 6) homogeneous part of f_1 (resp. f_2). It is clear however that the hypersurface singularity may be deformed locally so that the exceptional locus E_1 of the deformed singularity is a smooth del Pezzo surface of degree one in $\mathbf{P}(3,2,1,1)$. Note that the singularities can be resolved in the family by means of the weighted blow-up described above.

By taking a good representative of this deformation and applying Ehresmann's fibration theorem (with boundary), as was done in the proof of (1.1), we can find a neighbourhood U of E in X and a holomorphic deformation of the complex structure on U such that E deforms to a smooth del Pezzo surface of degree one.

The case $E^3 = 2$ is precisely analogous, with $\bar{X}$ having a hypersurface singularity with equation $x^2+f(y,z,t)=0$ with f a sum of monomials of degree ≥ 4, and the desingularization given by the weighted blow-up with weighting α given by $\alpha(x)=2$, $\alpha(y)=\alpha(z)=\alpha(t)=1$ (again see [12, (2.10) and (2.11)]). Thus E is naturally embedded in $\mathbf{P}(2,1,1,1)$ with equation $x^2+f_4(y,z,t)=0$. The previous argument applies again to show that by deforming the complex structure on a neighbourhood U of E, we can deform E to a smooth del Pezzo surface of degree two.

For $k = E^3 \geq 3$, we know that the morphism $\phi : X \to \bar{X}$ is an ordinary blow-up of the singular point, and $\bar{X}$ has local embedding dimension $k + 1$ [12]. In the case of $k = 3$ for instance, $\bar{X}$ still has a hypersurface singularity and E is a cubic hypersurface in $\mathbf{P}^3$. Again therefore, we can deform the complex structure on a neighbourhood U of E so that E deforms to a smooth cubic surface.

Lemma 2.1. *Suppose $\phi : X \to \bar{X}$ is a primitive Type II contraction of a surface E with $k = E^3 \leq 3$; then the Main Claim from §1 is true for the corresponding face of the nef cone.*

Proof. We can find a neighbourhood U of E and a holomorphic deformation of the complex structure on U such that E deforms to a smooth del Pezzo surface. Such a surface however has a finite number of (-1)-curves, and these are $(-1, -1)$-curves in the deformed complex structure on U. Let A be the homology class of one of these curves, a primitive class since $E \cdot A = -1$.

By patching this deformed complex structure on U together with the original one on X (as was done in (1.2)), we obtain an almost complex structure J' on X. Furthermore, we may assume that the above $(-1, -1)$-curves are the only pseudo-holomorphic rational curves on (X, J') representing A (the same argument via Gromov compactness as in the preamble to (1.2)), and that J' is regular for the class A. The proof is now similar to (1.2), showing that the 3-point Gromov-Witten invariant $\Phi_A(H, H, H)$ is positive and that A is J_1-effective for any J_1 as in the statement of the Main Claim. More precisely, the proof is a simpler version of (1.2) by reason of the fact that the homology class A is primitive, and so the Gromov-Witten invariant $\Phi_A(H, H, H)$ may be used.

Example 2.2. Consider the case where the exceptional surface E of the contraction $\phi : X \to \bar{X}$ is a projective cone on a smooth plane cubic. We can then deform the complex structure locally in a neighbourhood of E so that E deforms to a smooth cubic surface, containing 27 lines. The above argument shows that these 27 pseudo-holomorphic rational curves persist under almost complex deformations of the complex structure on X. A classical analysis of the degeneration of the 27 lines from the smooth cubic (see for instance [1]) reveals that they degenerate three at a time to the 9 generators of the cone E corresponding to the inflexion points of the plane cubic. Thus it is these generators which deform under an almost complex deformation of the structure on X, generically each deforming to three pseudo-holomorphic rational curves.

Lemma 2.3. *Suppose (V, P) is an isolated rational Gorenstein 3-fold singularity for which blowing up the point P yields a crepant resolution (U, E), with the exceptional divisor E irreducible and $E^3 = k > 3$. Then E is either a normal del Pezzo surface of degree $k \leq 9$ with at worst rational double point singularities, or a non-normal del Pezzo surface of degree $k = 7$ whose normalization is a non-singular rational ruled surface.*

Proof. We can reduce immediately to the case where E is non-normal; if E is normal, it cannot be an elliptic cone, since then E would have embedding dimension > 3 at its vertex, contradicting the assumption that U is smooth. The only remaining possibility in the normal case is that E is a del Pezzo surface of degree ≤ 9 with only rational double point singularities.

Suppose then that E is non-normal with $E^3 = k > 3$. Turning to the classification of non-normal del Pezzo surfaces in Theorem 1.1 of [14], we observe that case (c) (projection of a cone $\mathbf{F}_{k;0}$) does not occur, since the point of E corresponding to the vertex of $\mathbf{F}_{k;0}$ would have too high an embedding dimension. We are left therefore with the cases where E is either a projection $\bar{\mathbf{F}}_{a;1}$ of a scroll $\mathbf{F}_{a;1}$ ($k = a + 2$ where we assume $a \geq 2$), or the projection $\bar{\mathbf{F}}_{a;2}$ of a scroll $\mathbf{F}_{a;2}$ ($k = a + 4$ where we assume $a \geq 0$).

In the first case the projection of $\mathbf{F}_{a;1}$ is taken with centre a general point in the plane spanned by the minimal section and a fibre (both being lines), and the resulting surface $\bar{\mathbf{F}}_{a;1} \subset \mathbf{P}^{a+2}$ has a double line. In the second case, the projection of $\mathbf{F}_{a;2}$ is taken with centre a general point in the plane spanned by the minimal section (a conic). The resulting surface $\bar{\mathbf{F}}_{a;2} \subset \mathbf{P}^{a+4}$ again has a double line.

Let us consider the case first where $E \cong \bar{\mathbf{F}}_{a;2}$. It was observed first by Gross in [6] that this forces $a = 3$ and hence $k = 7$; we give here a different proof (part of which will be needed in §4) of this fact.

Let C denote the line of singularities of E and suppose its normal bundle is $N_{C/U} = (c, -2 - c)$, with the usual notation that $(r, s) = \mathcal{O}_{\mathbf{P}^1}(r) \oplus \mathcal{O}_{\mathbf{P}^1}(s)$. If $g : \tilde{U} \to U$ denotes the blow-up of U in C, we obtain on $\tilde{U}$ a scroll $E' \cong \mathbf{F}_e$ where $e = 2c + 2$. If A denotes the minimal section of E', we have $K_{E'} \sim -2A - (e + 2)f$ (where f denotes a fibre), and hence $-E'|_{E'} \sim A + (c + 2)f$. Moreover, if $\tilde{E}$ denotes the strict transform of E, it is easily checked that the morphism $\tilde{E} \to E$ coincides with the projection $\mathbf{F}_{a;2} \to \bar{\mathbf{F}}_{a;2}$, and hence that $\tilde{C} = \tilde{E} \cap E'$ is both isomorphic to $\mathbf{P}^1$ and is a double section of $E' \cong \mathbf{F}_e$. From this, an easy calculation shows that $\tilde{C} \sim 2A + (e+1)f$. But $\tilde{C}$ is irreducible and so $\tilde{C} \cdot A \geq 0$, from which we deduce $e \leq 1$, and hence (since e is even) that $e = 0$. Thus $c = -1$ and the normal bundle $N_{C/U} = (-1, -1)$.

To show that $a = 3$ and hence $k = 7$, we now calculate on $\tilde{E} \cong \mathbf{F}_{a;2}$. Note first that $(\tilde{C}^2)_{E'} = 4A^2 + 4(e+1) = 4$. Now

$$-2 = \deg K_{\tilde{C}} = (K_{\tilde{E}} + \tilde{C}) \cdot \tilde{C} = (\tilde{E} + 2E') \cdot \tilde{C} = 4 + 2(\tilde{C}^2)_{\tilde{E}}.$$

Thus $2(\tilde{C}^2)_{\tilde{E}} = -6$, i.e. $(\tilde{C}^2)_{\tilde{E}} = -3$. Hence $a = 3$ as claimed.

A similar elementary calculation shows that only $E^3 = 7$ could occur for the case $E \cong \bar{\mathbf{F}}_{a;1}$ with $a \geq 2$, but this in fact also follows from more general considerations. If the blow-up $(U, E) \to (V, P)$ is as in the statement of the Proposition (without any restriction on k), it is shown in the proof of Theorem 5.8 of [6] that any deformation E_1 of E may be achieved by deforming the complex structure on U, which may in turn be achieved by deforming the singularity (V, P). But for $a \geq 0$ the surface $\bar{\mathbf{F}}_{a+2;1}$ in $\mathbf{P}^{a+4}$ is a degeneration of surfaces isomorphic to $\bar{\mathbf{F}}_{a;2}$. To see this, we consider the scroll $\mathbf{F}_{a+1;2} \subset \mathbf{P}^{a+6}$ given parametrically by $(\lambda s^{a+3} : \lambda s^{a+2}t : \ldots : \lambda t^{a+3} : \mu s^2 : \mu st : \mu t^2)$, for $(\lambda : \mu) \in \mathbf{P}^1$ and $(s : t) \in \mathbf{P}^1$, and project from points $(0 : \ldots : 0 : u : 0 : 0 : v)$ on the ruling given by $s = 0$ to get a family of surfaces in $\mathbf{P}^{a+5}$. If we now project this family from the point $(0 : \ldots : 0 : -1 : 1 : 1) \in \mathbf{P}^{a+5}$, we get a family in $\mathbf{P}^{a+4}$ whose general member is a $\bar{\mathbf{F}}_{a;2}$, but which degenerates to a $\bar{\mathbf{F}}_{a+2;1}$ (corresponding to the first projection having centre the point $(0 : \ldots : 0 : 1)$ on $\mathbf{F}_{a+1;2}$). As remarked in [6], this fact together with the previous result shows that only the case $E \cong \bar{\mathbf{F}}_{5;1}$ can occur.

Proposition 2.4. *If $\phi : X \to \bar{X}$ is a primitive Type II contraction on a Calabi–Yau threefold X, then the Main Claim from §1 is true for the corresponding face of the nef cone.*

Proof. By (2.1), we may assume that $k = E^3 > 3$. Suppose first that E is non-normal, and so $E \cong \bar{\mathbf{F}}_{3,2}$ or $\bar{\mathbf{F}}_{5,1}$ in the notation of (2.3). It is easy to see ([6, (5.6)(iii)]) that $\bar{\mathbf{F}}_{3,2}$ can be smoothed in $\mathbf{P}^7$ to a smooth del Pezzo surface E_1 of degree 7, and hence by the argument in the proof of (2.3), the same is true for $\bar{\mathbf{F}}_{5,1}$. Moreover, we may apply again the argument from Theorem 5.8 of [6] to deduce that such a smoothing E_1 of E may be achieved by holomorphically deforming the complex structure on some neighbourhood U of E. The surface E_1 is a smooth del Pezzo surface of degree 7 containing precisely three $(-1,-1)$-curves. (We remark that in the case where $E \cong \bar{\mathbf{F}}_{3,2}$, one of these $(-1,-1)$-curves corresponds to the singular locus of E, which we saw in (2.3) was already a $(-1,-1)$-curve, and the other two correspond to the double fibres of $\bar{\mathbf{F}}_{3,2}$.) We let A be the homology class of such a $(-1,-1)$-curve. We may patch together the deformed holomorphic complex structure on U with the original complex structure J to get an almost complex structure J' on X, exactly as was done in (2.1). The proof of (2.1) now goes over unchanged to prove the Main Claim in this case.

By (2.3), we have reduced to the case where E is a del Pezzo surface with at worst rational double point singularities. Let us dispose first of the cases where E is neither $\mathbf{P}^2$, nor $\mathbf{P}^1 \times \mathbf{P}^1$, nor the quadric cone in $\mathbf{P}^3$. Here we can proceed in one of two ways. On the one hand, we can observe that E will contain lines with normal bundle $(-1,-1)$ or $(-2,0)$ in the threefold. In the latter case, the curves do not move in the threefold and so may be locally analytically contracted [13]. The argument from §1 then shows that such curves split up into disjoint $(-1,-1)$-curves when we make a local deformation of the complex structure, and the Main Claim follows as in (1.2). The alternative argument is the one we have just used above, where we observe that E may be smoothed, and that such a smoothing may be achieved by holomorphically deforming the complex structure on some neighbourhood U of E. Under such a deformation, E deforms to a smooth del Pezzo surface E_1, not isomorphic to either $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$, and hence containing $(-1,-1)$-curves. The proof from (2.1) now goes over unchanged to prove the Main Claim in these cases.

We consider now the cases where E is isomorphic to either $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$. In both cases, we have smooth families of $\mathbf{P}^1$s with no distinguished members, and we need to know how many of these curves will persist if we make a generic deformation of the almost complex structure on X. There is an intuitively attractive argument in §8 of [2] that this number is $(-1)^b e(B)$, where B is the smooth Hilbert scheme parametrizing the family, $e(B)$ the Euler characteristic, and b the dimension of B. The point is that the obstruction bundle to first order deformations is, as commented in [2], just the cotangent bundle to B, and a C^∞ 1-parameter deformation gives rise to a C^∞ 1-form on B; the above number is just the expected number of zeros of such a 1-form (taking into account orientations). The problem with this approach is that, unlike the algebraic case, it will not be enough to show that a given member of the family deforms to all finite orders; we will also have convergence to worry about. The way round this is to invoke a cobordism argument due to Ruan.

Consider the case where E is isomorphic to $\mathbf{P}^2$. We take A to be the homology class of a line on E – note that $E \cdot A = -3$. We now refer the reader to Proposition 5.7 of [15]. Observe that although A is not a primitive class in homology, it is both extremal in the cone of 1-cycles and not a non-trivial multiple of any J-effective class. Arguing from Gromov compactness, the same will be true for almost complex structures in some neighbourhood of J, and so for almost complex structures in such a neighbourhood we do not need to worry about multiple pseudo-holomorphic rational curves. The result from [15] does apply therefore to our case, even though A is not primitive. The other conditions of the Proposition are now easily checked to be satisfied, recalling that for $f : \mathbf{CP}^1 \to X$ a holomorphic rational curve, dim $coker(\bar{D}) = h^1(f^*T_X)$, and so the required equality holds because the Hilbert scheme B is smooth (in this case just $\mathbf{P}^2$). We deduce that for any generic compatible almost complex structure J' sufficiently close to J, the moduli space $\mathcal{M}_{A,J'}$ of non-parametrized pseudo-holomorphic rational curves is oriented cobordant to the zero set of a tranverse section of the cotangent bundle to B, and so consists of $e(B) = 3$ points when counted taking into account orientations. Thus for H a hyperplane class on (X, J), it follows as in (1.2) that the Gromov-Witten invariant $\tilde{\Phi}_A(H, H, H)$ is strictly positive, and so the Main Claim follows.

The case of E isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ is entirely analogous. If A denotes the class of a line on the quadric $\mathbf{P}^1 \times \mathbf{P}^1$, the corresponding Hilbert scheme consists of two disjoint lines, and in this case the Gromov-Witten invariant $\tilde{\Phi}_A(H, H, H)$ will be strictly negative. The remaining case, where E is isomorphic to the quadric cone, may be reduced to this case, if we smooth E by means of a local holomorphic deformation on some neighbourhood of E and patch together with the original complex structure J on X. The Main Claim is therefore proved for all Type II contractions.

3. Classification of Type III contractions

Proposition 3.1. *If $\phi : X \to \bar{X}$ is a primitive Type III contraction on a smooth Calabi–Yau threefold X, then the exceptional locus E is a* *conic bundle over a smooth curve C, where ϕ is just contraction along fibres on E.*

Proof. This is a natural generalization of an argument from pages 568-9 of [21], and we shall adopt the same notation as used there. In particular E will denote the exceptional divisor on X with C its image on $\bar{X}$; set $\pi : E \to C$ to be the induced morphism. Letting D denote the pullback of a hyperplane section of $\bar{X}$, we have for n sufficiently large that $nD - E$ is ample and that the map $H^0(X, \mathcal{O}_X(nD)) \to H^0(E, \mathcal{O}_E(nD))$ is surjective, from which it follows that $\pi_*\mathcal{O}_E = \mathcal{O}_C$. Under our assumptions, it was observed in [21] that the sheaf $\mathcal{O}_X(-E)$ is relatively generated by its global sections, i.e., that the map $\phi^*\phi_*\mathcal{O}_X(-E) \to \mathcal{O}_X(-E)$ is surjective. It follows as in [21, Theorem 2.2] that the map ϕ factors through the blow-up X_1 of X in C, and hence (from the primitivity of ϕ) if X_1 is normal, we may take ϕ to be the blow-up morphism. Furthermore the linear system $|nD - E|$ is without fixed points for n sufficiently large.

Once we have shown that C is smooth, an easy argument (cf. calculations below) then yields that X_1 is locally a complete intersection and non-singular in codimension one, and therefore normal; hence X is the blow-up of $\bar{X}$ in C. Since the intersection number of $-E$ with the general fibre of E over C is two, and all curves contained in fibres are in the same numerical ray, it follows (from the fact that C is a curve of cDV singularities) that E is a conic bundle over C. From this we see that the divisor $nD - E$ will in fact be very ample for n sufficiently large.

To show that C is smooth, we suppose that P is a point of C and Z is the corresponding reduced fibre over P; then Z consists of one or two components (in all cases isomorphic to $\mathbf{P}^1$). Taking S to be a general element of $|nD - E|$ for n sufficiently large, we observe that S can intersect Z in one or two points. The case where S intersects Z in two points Q_1 and Q_2 has been dealt with in [21]. For this case, we observe that S is smooth at both points, and the projection $\pi : E \to C$ yields a local analytic isomorphism between the curve $E \cap S$ at Q_i and the curve C at P. Thus we deduce that if P were a singular point of C, it would only have embedding dimension two. It is however observed on p. 569 of [21] that the blow-up X_1 of $\bar{X}$ in C is non-singular in codimension one and locally a complete intersection (by an argument given below, it is locally principal on $\tilde{\mathbf{C}}^4$), and hence normal, from which it follows that the blow-up must in fact be X. The calculation performed on p. 569 of [21] shows however (under the assumption that P has embedding dimension two) that this blow-up fails to be smooth when P is a singular point of C; we deduce therefore that C is smooth.

We are left with the case where the general element S of $|nD - E|$ intersects Z in one point, where Z is now isomorphic to $\mathbf{P}^1$. Setting $\tilde{C} = S \cap E$, we know that $\tilde{C}$ is a double cover of C with only a single point Q lying above P. The embedding dimension at Q of $\tilde{C}$ is at most two, and there is an involution $\iota : \tilde{C} \to \tilde{C}$ switching the two points of a general fibre. If moreover Q is a non-singular point of $\tilde{C}$, then P is a non-singular point of C; we shall assume from now on that Q is singular. Therefore locally analytically we have an induced involution $\iota : \mathbf{C}^2 \to \mathbf{C}^2$ with $\tilde{C}$ defined by an invariant (or anti-invariant) element $h \in \mathbf{C}[[\eta, \xi]]$.

The involution ι on $\mathbf{C}^2$ may be diagonalized so as to act via the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$; in the former case, the quotient is still $\mathbf{C}^2$ so C has embedding dimension two again and the previous argument from [21] holds. We should therefore concentrate on the latter case.

Let us assume first that $\tilde{C}$ is defined by an invariant element $h \in \mathbf{C}[[\eta, \xi]]$, whose terms are therefore all of even degree. We know that $P \in \bar{X}$ has embedding dimension 4; choose local coordinates x, y, z, t on $\mathbf{C}^4$ so that the quotient of S by the the induced involution ι has image given by $x = 0$, $yz = t^2$ (i.e., where $x = 0$ defines the smooth surface S on X and $t = \eta\xi$, $y = \eta^2$, $z = \xi^2$). The function h may now be considered as an element of $\mathbf{C}[[y, z, t]]$, and C is cut out by h on the surface given by $x = 0$, $yz = t^2$ in $\mathbf{C}^4$. The blow-up $\tilde{\mathbf{C}}^4$ of $\mathbf{C}^4$ in C may be considered (analytically) as the subvariety of $\mathbf{C}^4 \times \mathbf{P}^2$ defined by

$$\operatorname{rank} \begin{pmatrix} x & yz - t^2 & h \\ u & v & w \end{pmatrix} \leq 1,$$

where u, v, w are homogeneous coordinates on $\mathbf{P}^2$. Taking the affine piece of $\tilde{\mathbf{C}}^4$ given by $u = 1$, we get equations for this affine piece as a subvariety of $\mathbf{C}^6 = \mathbf{C}^4 \times \mathbf{C}^2$ to be $yz - t^2 = xv$, $h = xw$. Calculating the partial derivatives of the two equations and evaluating them at any point above $P = (0, 0, 0, 0) \in \mathbf{C}^4$, we get row vectors $(v, 0, 0, 0, 0, 0)$ and $(w, -a, -b, -c, 0, 0)$ where $h = ay + bz + ct +$ higher order terms. Let L denote the line in the locus $P \times \mathbf{P}^2 \subset \tilde{\mathbf{C}}^4$ given by $v = 0$; then the above calculation shows that $\tilde{\mathbf{C}}^4$ is singular at all points of L. Observe that $\tilde{\mathbf{C}}^4$ is non-singular in codimension one and locally a complete intersection, hence normal. Since the exceptional divisor on $\tilde{\mathbf{C}}^4$ is locally principal (from the fundamental property of blow-ups), it follows that the strict transform $X_1 \subset \tilde{\mathbf{C}}^4$ of $\bar{X} \subset \mathbf{C}^4$ is locally principal, where X_1 is of course also the blow-up of $\bar{X}$ in C. Thus X_1 is itself a local complete intersection. As argued before however, the map $\phi : X \to \bar{X}$ factors through this blow-up, and so X_1 is non-singular in codimension one and in particular normal. From the primitivity of ϕ, we deduce that X is isomorphic to X_1, i.e., X is just the blow-up of $\bar{X}$ in the curve C. Moreover, the exceptional locus of ϕ above P is the intersection of X with $P \times \mathbf{P}^2 \subset \tilde{\mathbf{C}}^4$, and this is just the fibre of E above P, i.e., in our case the curve Z. Thus X has a singularity where L meets Z in $\mathbf{P}^2$, contradicting our assumption that X was smooth.

The remaining case to consider is when the involution ι on $\mathbf{C}^2$ again acts via the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ but $\tilde{C}$ is defined by an anti-invariant element $h \in \mathbf{C}[[\eta, \xi]]$, whose non-zero terms are therefore all of odd degree. So $C \subset \mathbf{C}^4$ is defined by equations $x = 0$, $yz = t^2$, $h_1 = 0$, $h_2 = 0$, where $h_1, h_2 \in \mathbf{C}[[y, z, t]]$, and the images of h_1, h_2 in $\mathbf{C}[[\eta, \xi]]$ are respectively ηh and ξh. In particular, we have the relations

$$th_1 = yh_2 + (t^2 - yz)g_1, \quad \text{and} \quad th_2 = zh_1 + (t^2 - yz)g_2$$

for suitable $g_1, g_2 \in \mathbf{C}[[y, z, t]]$.

The blow-up $\tilde{\mathbf{C}}^4$ of $\mathbf{C}^4$ in C may then be considered as a component of the subvariety of $\mathbf{C}^4 \times \mathbf{P}^3$ defined by

$$\operatorname{rank} \begin{pmatrix} x & t^2 - yz & h_1 & h_2 \\ r & u & v & w \end{pmatrix} \leq 1,$$

where r, u, v, w are homogeneous coordinates on $\mathbf{P}^3$. Note that these equations define a reducible variety, with one component (which we do not want) having fibre $\mathbf{P}^3$ over all points of C. Observe now that the equations imply that $xtv = rth_1 = ryh_2 + r(t^2 - yz)g_1$, where $ryh_2 = xyw$. Thus $x(tv - yw) = r(t^2 - yz)g_1$. Multiplying by u, we deduce that

$$r(t^2 - yz)(tv - yw - ug_1) = 0,$$

and hence that $tv - yw - ug_1$ is in the ideal of definition for $\tilde{\mathbf{C}}^4 \subset \mathbf{C}^4 \times \mathbf{P}^3$. Similarly, we show that $tw - zv - ug_2$ is in the ideal of definition. Let $W \subset \mathbf{C}^4 \times \mathbf{P}^3$ be the subvariety defined by equations $xu = r(t^2 - yz)$, $tv - yw - ug_1 = 0$, $tw - zv - ug_2 = 0$, $xv = h_1$, $xw = rh_2$. Note that the fibre of W above a general point of C is $\mathbf{P}^2$, whilst above the point P it will be $\mathbf{P}^3$ when $g_1(P) = 0 = g_2(P)$, and $\mathbf{P}^2$ otherwise. Observe that if

$h = a\eta + b\xi +$ higher order terms, then $g_1(P) = b$ and $g_2(P) = a$; thus our assumption that Q is singular ensures that $g_1(P) = 0 = g_2(P)$.

Taking partial derivatives of the above set of defining equations, and evaluating the corresponding Jacobian matrix at any point of W above the origin $P \in \mathbf{C}^4$, it is easily checked that the corresponding matrix has rank 3 unless $v = w = 0$. Since the blow-up $\tilde{\mathbf{C}}^4$ coincides with W outside $P \times \mathbf{P}^3$, it follows that $W = \tilde{\mathbf{C}}^4$. The above calculation has shown that $\tilde{\mathbf{C}}^4$ has fibre $\mathbf{P}^3$ above P and is smooth except possibly along the line $v = w = 0$ in $P \times \mathbf{P}^3$. We consider the contraction $\bar{X} \subset \mathbf{C}^4$, a hypersurface containing C and having only cDV singularities along C – in fact generically along C either cA_1 or cA_2 singularities. Such a hypersurface will have a local analytic equation of the form

$$\begin{aligned}
0 = \;& F(x,y,z,t) \\
= \;& \alpha x^2 + \beta x(t^2 - yz) + \gamma(t^2 - yz)^2 + \delta_1 x h_1 + \delta_2 x h_2 \\
& +\sigma_1(t^2 - yz)h_1 + \sigma_2(t^2 - yz)h_2 + \text{terms involving } h_1^2, h_1h_2, h_2^2,
\end{aligned}$$

where $\alpha(P) \neq 0$ since F is not an element of m_P^3 (recalling that P is a cDV singularity).

If we look at the affine piece of the blow-up given by $r = 1$, the strict transform of $\bar{X}$ (i.e., the blow-up X_1) is given by an equation of the form

$$0 = \alpha + \beta u + \gamma u^2 + \delta_1 v + \delta_2 w + \sigma_1 uv + \sigma_2 uw + \text{terms involving } v^2, vw, w^2.$$

So X_1 is cut out on $\tilde{\mathbf{C}}^4$ by a single extra equation, which therefore cuts out a two-dimensional subvariety on $P \times \mathbf{P}^3$, not containing the line $v = w = 0$ in $P \times \mathbf{P}^3$.

Under the assumption that P is a singular point of C, the above analysis has shown that the fibre of X_1 above P is two-dimensional. As observed before, $\phi : X \to \bar{X}$ factors through the blow-up X_1 of $\bar{X}$ in C. Since the fibre of X above P is 1-dimensional, we arrive at the required contradiction. The proof of (3.1) is now complete.

Having seen that C is smooth, we can make a local calculation to see what are the possible dissident singularities. In the case where the generic singularity is an cA_1 singularity, the restrictions obtained are rather weak (see [21, Section 2], for a discussion of this case). In the case where the generic singularity is a cA_2 singularity, the restrictions on a dissident singularity P (forced by the fact that X is smooth) are far stronger.

Proposition 3.2. *If the singularity at the generic point of C is a* cA_2 *singularity, then any dissident singularity P will be a* cD_4 *or* cE_r *singularity ($r = 6, 7, 8$). Moreover, the exceptional divisor E on X has only a pinch point singularity on the fibre above a dissident point.*

Proof. This is just a question of checking the various cases. If for instance P were a cA_n singularity ($n \geq 3$), then we could choose local analytic coordinates (x, y, z, t) at P so that C is given by $x = y = z = 0$ and $\bar{X}$ is locally given by an equation of the form $x^2 + y^2 + z^{n+1} + tg(x, y, z, t) = 0$. Our assumptions imply that all terms of g are at least quadratic in x, y, z and that there is no term in z^2 only. Therefore, if we blow up the line $x = y = z = 0$, we obtain an affine piece of X with equation

$$0 = x^2 + y^2 + z^{n-1} + \text{other quadratic or higher degree terms in } x, y, z, t,$$

contradicting the assumption that X was smooth.

We now consider the case of a cD_n singularity ($n \geq 4$). Again, we choose local coordinates so that C is given by $x = y = z = 0$ and $\bar{X}$ is locally given by an equation of the form $x^2 + y^2 z + z^{n-1} + tg(x, y, z, t) = 0$, where the terms of g are at least quadratic in x, y, z. If $n > 4$, there must be a z^2 term in g (otherwise there is an obvious affine piece of the blow-up which is singular), but no term in y^2 (since for $t \neq 0$ we have a cA_2 singularity). We then check that a different affine piece of X is still singular. In the case where $n = 4$, we have the alternative that g contains a term in y^2 but no term in z^2, and so the equation takes the form $0 = x^2 + y^2 z + z^3 + ty^2 +$ further terms. Moreover, by taking the local equation for $\bar{X}$ as $0 = x^2 + y^2 z + z^3 + ty^2$, we see this case cannot be ruled out by local arguments. A relevant affine piece of the blow-up X is then given by an equation

$$0 = x^2 + y^2 z + z + ty^2 + \text{terms involving } z,$$

and so E (given locally by $z = 0$) has only the singularity claimed on the dissident fibre.

A similar argument works for the cE_r singularities ($r = 6, 7, 8$). Again we choose local coordinates so that C is given by $x = y = z = 0$ and $\bar{X}$ is locally given by an equation of the form $h(x, y, z) + tg(x, y, z, t) = 0$, where $h = x^2 + y^3 + z^4$ for $r = 6$, $h = x^2 + y^3 + yz^3$ for $r = 7$, and $h = x^2 + y^3 + z^5$ for $r = 8$, and where the terms of g are at least quadratic in x, y, z. Similar arguments to those above force the existence of a z^2 term in g (and by taking the local equation for $\bar{X}$ to be $0 = h + tz^2$, we see that the case of cE_r singularities cannot be ruled out). Again however, by blowing up we see that E has only a pinch point singularity (of the form $x^2 + tz^2 = 0$) on the dissident fibre.

4. Proof of Main Claim for Type III contractions

Suppose now that we have a codimension-one face of the Kähler cone of (X, J) corresponding to a Type III contraction $\phi : X \to \bar{X}$. Let E denote the exceptional divisor, which from the results of the previous Section we know to be a conic bundle over a smooth curve C.

We now observe that if C has genus $g > 0$, the surface E will not deform under a generic holomorphic deformation of (X, J), and moreover will only deform over a locus in the Kuranishi space of codimension at least g. As it is sufficient to prove it for first order deformations, this can be proved using a generalization of the Hodge theoretic arguments from [21]. It will however also follow from the fact that if (X, J) were to contain such a conic bundle E over C, there would be a Abel-Jacobi map from the Jacobian $J(C)$ to the intermediate Jacobian $J(X)$, a principally polarized analytic torus, which is an inclusion (cf. §4 of [21]). Considering the period map, it follows from the unobstructedness theorem of Bogomolov, Tian and Todorov that it is a codimension g condition in moduli for $J(X)$ to contain the g-dimensional abelian variety $J(C)$.

Thus if $g > 0$, only finitely many fibres of E will deform under the generic deformation of (X, J). We can however also observe that the converse is true, that if $g = 0$ then E will deform for all holomorphic deformations of (X, J). This follows from the fact that E is Gorenstein with $h^1(\mathcal{O}_E) = 0$. From the exact sequence

$$0 \to \mathcal{O}_X(-E) \to \mathcal{O}_X \to \mathcal{O}_E \to 0$$

we obtain $h^i(X, \mathcal{O}_X(E)) = 0$ for $i > 0$, and hence $\chi(\mathcal{O}_X(E)) = 1$. As argued in §2 of [21], it then follows from the Base Change Theorem that the divisor class E remains effective when we deform the complex structure, and so the surface E deforms as claimed.

We concentrate first on the case where the general fibre of E over C is irreducible (i.e., isomorphic to $\mathbf{P}^1$).

When $g > 0$, we can apply the arguments from [21] directly. If E has a singular fibre, then the fibre contains a curve $l \cong \mathbf{P}^1$ with $E \cdot l = -1$ and l not moving in X. A standard Hilbert scheme argument (see [21,

§3]) then shows that l must deform under any holomorphic deformation of the complex structure on X. If E contains no singular fibres, then it is a ruled surface over C. A straightforward dimension counting argument (Proposition 4.2 from [21]) yields that when E is ruled over a curve of genus $g > 1$, some fibres of E do deform under a holomorphic deformation of (X, J). Thus, unless E is ruled over an elliptic curve, the given codimension-one face of the Kähler cone corresponds to a Type I contraction on the generic holomorphic deformation, contracting finitely many curves isomorphic to $\mathbf{P}^1$. The Main Claim therefore follows from the Type I case, as proved in §1.

To conclude the argument for the case where E has irreducible generic fibre, we need to consider the case $g = 0$. This was not a problem in [21], since as commented above the whole surface deforms under a holomorphic deformation of the complex structure on X. This does not however tell us what happens when we deform to an almost complex structure on X.

The case where E is a ruled surface over $\mathbf{P}^1$ may be dealt with by the cobordism argument from [15] which we outlined at the end of §2. If A is the class of a fibre of E, then the corresponding Hilbert scheme is isomorphic to $\mathbf{P}^1$, and so in this case the three-point Gromov-Witten invariant is strictly negative and the Main Claim follows. We remark in passing that this argument gives another proof of the Main Claim for the case where E is ruled over a curve C of genus $g > 1$, since in that case the Gromov-Witten invariant is strictly positive.

For the case where C has genus 0 but E has singular fibres, we need to replace the Hilbert scheme argument referred to above by a local argument using the same tools as were used for Type I contractions. This will of course work for an arbitrary genus base curve, and so (given the remark in the previous paragraph) we can if we wish avoid completely the argument of deforming to a Type I contraction. Let Z denote a dissident fibre of E lying over $Q \in C$, so that Z is either a line pair or a double line.

Lemma 4.1. *We can find an open disc $Q \in \Delta \subset C$ and an open neighbourhood U of Z in X with $U \to \Delta$ a differentiably trivial family containing the fibre of E over P for all $P \in \Delta$, and a holomorphic deformation of the complex structure on U for which the dissident fibre Z deforms into at least two disjoint $(-1,-1)$-curves, but no other fibre deforms.*

Proof. This is very nearly a repetition of the proof of (1.1). Let

$\phi : X \to \bar{X}$ be the contraction; then $\bar{X}$ is locally just a family of rational double point surface singularities, parametrized by a disc $\Delta \subset C$ with centre Q. We therefore have a neighbourhood U of Z in X with $U \to \Delta$ and fibre U_0 over Q containing the dissident fibre Z.

In the notation of [4, p. 678], we have a map $g : \Delta \to S$ so that the diagram

$$\begin{tikzcd} \Delta \arrow[r] \arrow[dr] & S \arrow[d, "\pi"] \\ & R \end{tikzcd}$$

commutes, where $R = \mathrm{Def}\ \bar{U}_0$ and $S = \mathrm{Def}\ U_0$.

We may now holomorphically deform the family $U \to \Delta$ by means of deforming g, first so that the image of Q is unchanged but no other point $P \in \Delta$ has image in $D = \pi^{-1}$(discriminant locus), and then so that the map is tranverse to D. In this way we obtain (shrinking Δ if necessary) a deformation $\mathcal{U} \to \Delta \times \Delta'$ of $U \to \Delta$, where for $t \neq 0$, the fibre Z deforms to at least two disjoint $(-1,-1)$-curves, and no other fibre of $E_1 = E|_\Delta$ deforms (i.e., we resolve the singularities of the corresponding family $\bar{\mathcal{U}} \to \Delta$, where for $t \neq 0$, the threefold $\bar{\mathcal{U}}_t$ has singular locus consisting of $\delta > 1$ nodes).

Since $\Delta \times \Delta'$ is smooth, we may take a good representative $\bar{\mathcal{U}}$ of the family over $\Delta \times \Delta'$ by Theorem 2.8 of [9], and hence deduce as in (1.1) that $\mathcal{U}$ is differentiably trivial over $\Delta \times \Delta'$. Taking $U = \mathcal{U}_0$, the Lemma is proved.

In order to complete our proof of the Main Claim for the case where the generic fibre of E over C is irreducible, we need the following corollary of (4.1).

Proposition 4.2. *The Main Claim is true in the Type III case where E has irreducible generic fibre but is not ruled over C.*

Proof. Let A be the homology class of an irreducible component of any singular fibre Z; we remark that $E \cdot A = -1$ and that A is primitive. As in (4.1), we can find an open neighbourhood U of Z with $U \to \Delta$ and $E_1 = E|_\Delta$ having only the one singular fibre Z (above $Q \in \Delta$), and a holomorphic deformation of the complex structure on U so that Z splits into at least two $(-1,-1)$-curves. We may do this for each of the singular fibres $Z_1 = Z, Z_2, \ldots Z_N$, obtaining deformed holomorphic structures on each of the corresponding neighbourhoods $U_1 = U, \ldots, U_N$. These may be patched together with the original complex structure J on X to give an almost complex structure J' on X, with respect to which the primitive class A is represented by a finite number of $(-1,-1)$-curves lying in the open sets $U_1, \dots, U_N$. Moreover, as in the proof of (1.1) we may assume that these are the only pseudo-holomorphic rational curves representing the class A. This latter statement follows as in (1.1) from Gromov compactness, since by considering the compact subset F of X given by the complement of suitably small open neighbourhoods of the singular fibres Z_i, we may assume that any other pseudo-holomorphic rational curve on (X, J') representing A intersects F. If this were true for all the almost complex structures J' constructed in the above way however near to J they were, we would deduce from Gromov compactness that there was a holomorphic curve on (X, J) representing A and intersecting F, which is not the case.

We remark in passing that it is this step which fails when E is a ruled surface (say over an elliptic curve C). Here, we have to take A to be the homology class of a fibre, and then by local holomorphic deformations we can always, in a similar way to (4.1), force certain specified fibres to deform to $(-1,-1)$-curves, which therefore count positively towards the Gromov-Witten invariant. We can however no longer say that these are the only pseudo-holomorphic rational curves representing the class A, since any fibre of E could be the limit of pseudo-holomorphic rational curves representing A for J' tending to J. In the case of C elliptic, there must of course be pseudo-holomorphic rational curves on (X, J') which count negatively towards the Gromov-Witten invariant and precisely cancel out the positive contributions.

In the case under consideration however, we argue as in before that the Gromov-Witten invariant $\Phi_{A,J'}(H, H, H)$ is strictly positive, and hence the Main Claim follows.

Having proved the Main Claim under the assumption that the generic fibre of E is irreducible, we assume from now on that the generic fibre of E over C is a line pair. In this case, we have a family of lines on X parametrized by a smooth curve $\tilde{C}$, namely the double cover of C branched over the points of C whose fibres are double lines. The Hilbert scheme corresponding to the lines will be just $\tilde{C}$ with embedded components at the dissident points. We prove the Main Claim first under the additional assumption that $g(\tilde{C}) > 0$.

We observe that a general line in the family will deform over a locus in the Kuranishi space for X of codimension at most one (a standard argument – see (3.1) of [21]). Let $\tilde{E}$ denote the normalization of E (a ruled surface over $\tilde{C}$) and $f : \tilde{E} \to X$ the induced map to X. I claim that the map $H^1(T_X) \to H^1(\omega_{\tilde{E}})$ (i.e., $H^{2,1}(X) \to H^{2,1}(\tilde{E})$) is surjective. This follows from Hodge theory and the fact that the contraction $\bar{X}$ of X has only quotient singularities in codimension 2, by an entirely analogous argument to that of (4.1) in [21]. Observe now that the above map factors as $H^1(T_X) \to H^1(f^*T_X) \to H^1(N_f) \to H^1(\omega_{\tilde{E}})$, and so the obstruction to the morphism f deforming to first order gives at least a codimension $g(\tilde{C})$ condition in $H^1(T_X)$, i.e., f deforms over a locus in the Kuranishi space of codimension at least $g(\tilde{C})$. This fact also follows by consideration of the intermediate Jacobian $J(X)$, since under the above circumstances there is an Abel-Jacobi inclusion map of the Jacobian $J(\tilde{C})$ into $J(X)$.

So, still under the assumption that $g(\tilde{C}) > 0$, we have various cases. If $g(C) = 0$, then E will deform under all complex deformations of (X, J), and under the generic deformation it must deform to a rational conic bundle with irreducible generic fibre; this case has already been covered. Let us assume therefore that we also have $g(C) > 0$. If $g(\tilde{C}) > 1$, the facts stated in the previous paragraph and a dimension counting argument on the Hilbert scheme of lines in the versal deformation $\mathcal{X} \to B$ show that some line will always deform (cf. Erratum and (3.1) of [21]). Since E does not deform for the generic deformation, it follows that on such a deformation we have a Type I contraction, and our Main Claim follows from the results of §1. If however $g(\tilde{C}) = 1$, then our assumptions imply that $g(C) = 1$ and the map $\tilde{C} \to C$ is unramified – this then is the case of E being a quasi-ruled surface over an elliptic curve, which because of the results of the Erratum to [21] has been excluded when stating the Main Claim.

The only other possibility left is the case that $g(\tilde{C}) = 0$; here, C is also rational and the double cover is ramified over two points (i.e,. E has two double fibres). If C_0 denotes the line of singularities of E, the blow-up of E in C_0 will just be the normalization $\tilde{E}$, a rational scroll (cf. (3.2)). The calculations performed in the proof of (2.3) apply equally well in this case to show that C_0 is a $(-1,-1)$-curve and that E is isomorphic to the non-normal del Pezzo surface $\bar{\mathbf{F}}_{3,2}$. In particular, E may be contracted locally analytically to an isolated Gorenstein singularity. We now argue as in (2.4) that by suitably deforming the holomorphic structure locally on a neighbourhood U of E, we may smooth E to a non-singular del Pezzo surface of degree 7, therefore containing three $(-1,-1)$-curves. In the Type III case however, these curves no longer all have the same numerical class, in that the class corresponding to the singular locus of E is different from that of the other two $(-1,-1)$-curves. If we choose our homology class A to be the class of one of these other two $(-1,-1)$-curves, then A will define the codimension-one face of $\mathcal{K}$ that we want. Moreover, the argument from (2.4) can be applied with the primitive class A to give the positivity of the Gromov-Witten invariant $\Phi_A(H,H,H)$, and hence to prove the Main Claim in this case.

The Main Claim has therefore now been proved for all the three types of primitive contraction.

Acknowledgments

Much of this work was completed by the author whilst a Visiting Scholar for a month at Harvard University in Spring 1995, and he would like to thank Prof. S.-T. Yau for making this visit possible. Thanks are also due to Mark Gross, Sheldon Katz and Dusa McDuff for valuable conversations and comments at various stages of the work.

References

[1] J.W. Bruce & C.T.C. Wall, *On the classification of cubic surfaces*, J. London Math. Soc. **19** (1979) 245-256.

[2] P. Candelas, A. Font, D.R. Morrison & S. Katz, *Mirror symmetry for two parameter models - II*, Nuclear Phys. B **429** (1994) 626-674.

[3] M. Demazure, *Surfaces de Del Pezzo*, Seminaire sur les singularités des surfaces (M. Demazure, H. Pinkham and B. Teissier, eds.), Lecture Notes in Math. Vol. 777, Springer, Berlin, 1980, 23-70.

[4] R. Friedman, *Simultaneous resolution of threefold double points*, Math. Ann. **274** (1986) 671-689.

[5] M. Gross, *The deformation space of Calabi–Yau n-folds with canonical singularities can be obstructed*, Mirror symmetry, II (B. Greene and S.-T. Yau, eds.), Stud. Adv. Math., Vol 1, Amer. Math. Soc. & Internat. Press, 1996, 401-411.

[6] ———, *Deforming Calabi–Yau threefolds*, to appear in Math. Ann, 1997.

[7] F. Hidaka & K. Watanabe, *Normal Gorenstein surfaces with ample anti-canonical divisor*, Tokyo J. Math. **4** (1981) 319-330.

[8] M. Kontsevich, *Homological algebra of mirror symmetry*, Proc. ICM, Zurich 1994, Birkhäuser, Basel, 1995, 120-139.

[9] E. J. N. Looijenga, *Isolated singular points on complete intersections*, Cambridge Univ. Press, 1984.

[10] D. McDuff, *Elliptic methods in symplectic geometry*, Bull. Amer. Math. Soc. **23** (1990) 311-358.

[11] D. McDuff & D. Salamon, *J-holomorphic curves and quantum cohomology*, Univ. Lecture Series, Vol. 6, Amer. Math. Soc. 1994.

[12] M. Reid, *Canonical 3-folds*, Géométrie algébrique, (A. Beauville ed.), Angers 1979, Sijthoff and Noordhoff, The Netherlands, 1990, 273-310.

[13] ———, *Minimal models of canonical 3-folds*, Algebraic varieties and analytic varieties, (S. Iitaka ed.), Adv. Stud. Pure Math. Vol. 1, Kinokuniya, Tokyo and North Holland, Amsterdam, 1983, 395-418.

[14] ———, *Nonnormal del Pezzo surfaces*, Publ. Res. Inst. Math. Sci., Kyoto, **30** (1994) 695-727.

[15] Y. Ruan, *Symplectic topology and extremal rays*, Geom. Funct. Anal. **3** (1993) 395-430.

[16] ———, *Topological sigma model and Donaldson type invariants in Gromov theory*, to appear in Duke J. Math.

[17] Y. Ruan & G. Tian, *A mathematical theory of quantum cohomology*, J. Differential Geom. **42** (1995) 259-367.

[18] J.-C. Sikorav, *Some properties of holomorphic curves in almost complex manifolds*, Holomorphic curves in symplectic geometry, (M. Audin and J. Lafontaine eds), Ch. V, Birkhäuser, Progr. Math. Vol. 117, Basel-Boston, 1994, 165-190.

[19] C. T. C. Wall, *Classification problems in differential topology V.* On certain 6-manifolds, Invent. Math. **1** (1966) 355-374.

[20] P. M. H. Wilson, *Calabi–Yau manifolds with large Picard number*, Invent. Math. **98** (1989) 139-155.

[21] ———, *The Kähler cone on Calabi–Yau threefolds*, Invent. Math. **107** (1992) 561-583; *Erratum*, Invent. Math. **114** (1993) 231-233.

[22] ———, *Elliptic ruled surfaces on Calabi-Yau threefolds*, Math. Proc. Cambridge Philos. Soc. **112** (1992) 45-52.

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